Solutions for Mathematical Reflections 5(2006)

Juniors

J25. Let k be a real number different from 1. Solve the system of equations

$$\begin{cases} (x+y+z)(kx+y+z) = k^3+2k^2 \\ (x+y+z)(x+ky+z) = 4k^2+8k \\ (x+y+z)(x+y+kz) = 4k+8 \ . \end{cases}$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

First solution by José Luis Díaz-Barrero and José Gibergans-Báguena, Universitat Politècnica de Catalunya, Barcelona, Spain.

Solution. Setting $s = x+y+z$ and adding up the three equations given, we obtain

$$s(kx+2x+ky+2y+kz+2z) = k^3+6k^2+12k+8,$$

$$(x+y+z)(k+2) = (k+2)^3,$$

and

$$s = \pm(k+2).$$

If $x+y+z=0$, then $k=-2$, also if $k=-2$ we get $x=y=z=0$.

Otherwise we distinguish the cases (i) when $s=k+2$ and (ii) when $s=-(k+2)$.

(i) If $s=(k+2)$, then

$$\begin{cases} (k+2)(kx+y+z) &= k^2(k+2) \\ (k+2)(x+ky+z) &= 4k(k+2) \\ (k+2)(x+y+kz) &= 4(k+2) \end{cases},$$

or equivalently

$$\begin{cases} kx+y+z &= k^2 \\ x+ky+z &= 4k \\ x+y+kz &= 4 \end{cases}$$

and using $x+y+z=k+2$ we get

$$x = \frac{(k-2)(k+1)}{k-1}, y = \frac{3k-2}{k-1}, z = \frac{-(k-2)}{k-1}.$$

(ii) If $s = -(k+2)$, then

$$x = -\frac{(k-2)(k+1)}{k-1}, y = -\frac{3k-2}{k-1}, z = \frac{k-2}{k-1}$$

is the solution obtained. Notice that in both cases we have $k \neq 1$, as stated, and we are done.

Second solution by Ashay Burungale, India.

Solution. We observe that $x + y + z = 0$ forces $k = -2$.
The case $k = -2$ forces $kx + y + z = x + ky + z = x + y + kz = 0$, which gives us $x = y = z = 0$. Assume that $x + y + z$ to be nonzero and k different from -2.

Dividing the third equation by the second, we get

$$\frac{x + ky + z}{x + y + kz} = k, \text{ and thus } x(k-1) = z(1-k^2).$$

As $k \neq 1$, it follows that $x = -(k+1) \cdot z$. (1)

Dividing the first equation by the second, we get

$$\frac{kx + y + z}{x + ky + z} = \frac{k}{4}, \text{ and thus } z(k-4) + y(k^2-4) = 3kx.$$

Using first relation (1) we have

$$z(k-4) + y(k^2-4) = -3k(k+1)z,$$

$$y(k^2-4) = z(-3k^2-4k+4),$$

$$y(k-2)(k+2) = -z(3k-2)(k+2).$$

Thus we have $y = -\frac{3k-2}{k-2} \cdot z$. (2)

Plugging results (1) and (2) in the third equation, we get

$$z^2(-(k+1) - \frac{3k-2}{k-2} + 1)(-(k+1) - \frac{3k-2}{k-2} + k) = 4(k+2),$$

$$z^2(k^2+k-2)(4(k-1)) = 4(k+2)(k-2)^2.$$

Therefore $z = \mp\frac{k-2}{k-1}$ and $x = \pm\frac{(k-2)(k+1)}{k-1},\ y = \pm\frac{3k-2}{k-1}$.

J26. A line divides an equilateral triangle into two parts with the same perimeter and having areas S_1 and S_2, respectively. Prove that

$$\frac{7}{9} \leq \frac{S_1}{S_2} \leq \frac{9}{7}$$

Proposed by Bogdan Enescu, "B.P. Hasdeu" National College, Romania

First solution by Vishal Lama, Southern Utah University.

Solution. Without loss of generality, we may assume that the given equilateral triangle ABC has sides of unit length, $AB = BC = CA = 1$. If the line cuts the triangle in two triangles them clearly $\frac{S_1}{S_2} = 1$.

We may assume that the line cuts side AB at D and AC at E. Let the area of triangle $ADE = S_1$ and the area of quadrilateral $BDEC = S_2$.

Then, $S_1 + S_2 =$ area of equilateral triangle $ABC = \frac{\sqrt{3}}{4}$.

Let $BD = x$ and $CE = y$. Then, $AD = 1 - x$ and $AE = 1 - y$. Since the regions with areas S_1 and S_2 have equal perimeter, we must have $BD + BC + CE = AD + AE$.

$$x + 1 + y = (1 - x) + (1 - y), \quad \Rightarrow x + y = \frac{1}{2}.$$

Now, area of triangle $ADE = S_1 = \frac{1}{2} \cdot AD \cdot AE \cdot \sin(\angle DAE)$,

$$S_1 = \frac{1}{2}(1 - x)(1 - y)\sin 60°, \quad \Rightarrow S_1 = \frac{\sqrt{3}}{4}(1 - x)(\frac{1}{2} + x).$$

Denote $a = \frac{S_2}{S_1} > 0$, we get that

$$\frac{S_1}{S_1 + S_2} = \frac{1}{1 + a} = (1 - x)(\frac{1}{2} + x),$$

which after some simplification yields

$$2x^2 - x + \frac{1 - a}{1 + a} = 0.$$

The above quadratic equation in x has real roots and the discriminant should be greater or equal to zero. Thus

$$\Delta = 1 - 4 \cdot 2 \cdot \left(\frac{1 - a}{1 + a}\right) = \frac{9a - 7}{a + 1} \geq 0.$$

Therefore $a \geq \frac{7}{9}$ or $\frac{S_2}{S_1} \geq \frac{7}{9}$. Changing our the notations: area of triangle $ADE = S_2$ and area of quadrilateral $BDEC = S_1$ we get that $\frac{S_1}{S_2} \geq \frac{7}{9}$. Thus

$$\frac{7}{9} \leq \frac{S_1}{S_2} \leq \frac{9}{7}.$$

Second solution by Daniel Campos Salas, Costa Rica.

Solution. Suppose without loss of generality, that the triangle has sidelength 1. Note that this implies $S_1 + S_2 = \dfrac{\sqrt{3}}{4}$. The line can divide the triangle into a triangle and a quadrilateral or two congruent triangles. The second case is obvious. Since the inequality is symmetric with respect to S_1 and S_2 we can assume that S_2 is the area of the new triangle.

Let l be one of the sides of the new triangle which belongs to perimeter of the equilateral triangle. The other side of the new triangle in the perimeter equals $\left(\dfrac{3}{2} - l\right)$. Then, $S_2 = l\left(\dfrac{3}{2} - l\right)\dfrac{\sqrt{3}}{4}$. Note that the inequality is equivalent to

$$\frac{16}{9} \leq \frac{S_1 + S_2}{S_2} \leq \frac{16}{7}, \text{ or}$$

$$\frac{7}{16} \leq l\left(\frac{3}{2} - l\right) \leq \frac{9}{16}. \tag{1}$$

From the inequality $\left(l - \dfrac{3}{4}\right)^2 \geq 0$, it follows that $l\left(\dfrac{3}{2} - l\right) \leq \dfrac{9}{16}$, and this proves the RHS inequality of (1). Since l and $\left(\dfrac{3}{2} - l\right)$ are smaller than the equilateral triangle sides it follows that $l, \left(\dfrac{3}{2} - l\right) \leq 1$, that implies that $l \in \left[\dfrac{1}{2}, 1\right]$. Now, the LHS inequality of (1) is equivalent to

$$0 \geq 16l^2 - 24l + 7,$$

which holds if and only if $l \in \left[\dfrac{3 - \sqrt{2}}{4}, \dfrac{3 + \sqrt{2}}{4}\right]$, which is true because $\dfrac{3 - \sqrt{2}}{4} < \dfrac{1}{2}$ and $1 < \dfrac{3 + \sqrt{2}}{4}$, and we are done.

J27. Consider points M, N inside the triangle ABC such that $\angle BAM = \angle CAN, \angle MCA = \angle NCB, \angle MBC = \angle CBN$. M and N are isogonal points. Suppose $BMNC$ is a cyclic quadrilateral. Denote T the circumcenter of $BMNC$, prove that $MN \perp AT$.

Proposed by Ivan Borsenco, University of Texas at Dallas

First solution by Aleksandar Ilic, Serbia.

Solution. As T is circumcenter of quadrilateral $BMNC$, we have $TM = TN$. We will prove that $AN = AM$, and thus get two isosceles triangles over base MN meaning $AT \perp MN$. We have to prove that $\measuredangle ANM = \measuredangle AMN$. Because $BMNC$ is cyclic quadrilateral we have $\measuredangle MCN = \measuredangle NBM$. Let's calculate angles:

$$\measuredangle ANM = 360^o - (\measuredangle CNM + \measuredangle ANC) = \measuredangle CBM + \measuredangle ACN + \measuredangle CAN.$$

$$\measuredangle AMN = 360^o - (\measuredangle BMN + \measuredangle AMB) = \measuredangle BCN + \measuredangle ABM + \measuredangle BAM.$$

We know that $\measuredangle CAN = \measuredangle BAM$.

From the equality $\measuredangle BCN + \measuredangle ABM = (\measuredangle BCM + \measuredangle MCN) + \measuredangle ABM = \measuredangle ACN + (\measuredangle MBN + \measuredangle NBC) = \measuredangle ACN + \measuredangle CBM$ we conclude that $\measuredangle ANM = \measuredangle AMN$.

Second solution by Prachai K, Thailand.

Solution. Using Sine Theorem we get

$$\frac{AM}{\sin \angle ABM} = \frac{BM}{\sin \angle BAM}, \ \frac{AN}{\sin \angle ACN} = \frac{CN}{\sin \angle CAN}.$$

As $\angle BAM = \angle CAN$ we have

$$\frac{AM}{AN} = \frac{BM \cdot \sin \angle ABM}{CN \cdot \sin \angle ACN} = \frac{2R \cdot \sin \angle BCM \cdot \sin \angle ABM}{2R \cdot \sin \angle CBN \cdot \sin \angle ACN}.$$

Using the fact that $\angle BCM = \angle ACN$ and $\angle CBN = \angle ABM$ we get

$$\frac{AM}{AN} = \frac{\sin \angle ACN \cdot \sin \angle ABM}{\sin \angle ABM \cdot \sin \angle ACN} = 1.$$

Clearly the perpendiculars form A and T to MN both bisect MN, it follows that $AT \perp MN$.

Also solved by Ashay Burungale, India.

J28. Let p be a prime such that $p \equiv 1(\bmod 3)$ and let $q = \lfloor \frac{2p}{3} \rfloor$. If

$$\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(q-1)q} = \frac{m}{n}$$

for some integers m and n, prove that $p|m$.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

First solution by Aleksandar Ilic, Serbia.

Solution. Let $p = 3k+1$ and $q = \lfloor \frac{2p}{3} \rfloor = 2k$. When considering equation modulo p, we have to prove that it is congruent with zero mod p.

$$S = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{(q-1)\cdot q} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{q-1} - \frac{1}{q}.$$

Now regroup fractions, and substitute $q = 2k$.

$$S = \sum_{i=1}^{q} \frac{1}{i} - 2\sum_{i=1}^{q/2} \frac{1}{2i} = \sum_{i=1}^{2k} \frac{1}{i} - \sum_{i=1}^{k} \frac{1}{i}.$$

From Wolstenholme's theorem we get that:

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1} \equiv 0(\bmod p^2).$$

Because $-i \equiv_p p - i$, we have:

$$S = \sum_{i=1}^{p-1} \frac{1}{i} - \sum_{i=2k+1}^{p-1} \frac{1}{i} + \sum_{i=1}^{k} \frac{1}{p-i} \equiv_p 0 - \sum_{i=2k+1}^{3k} \frac{1}{i} + \sum_{i=1}^{k} \frac{1}{3k+1-i} \equiv 0(\bmod p).$$

Second solution by Ashay Burungale, India.

Solution. Note that $p = 1(\bmod 6)$. Let $p = 6k+1$, thus $q = \lfloor \frac{2p}{3} \rfloor = 4k$. We have

$$\frac{m}{n} = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{(q-1)\cdot q} = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(4k-1)\cdot 4k} =$$

$$1 + \frac{1}{3} + \ldots + \frac{1}{4k-1} - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{4k}\right) = \frac{1}{2k+1} + \frac{1}{2k+2} + \ldots + \frac{1}{4k}.$$

Grouping $\left(\frac{1}{2k+1}, \frac{1}{4k}\right)$, $\left(\frac{1}{2k+2}, \frac{1}{4k-1}\right)$,..., $\left(\frac{1}{3k}, \frac{1}{3k+1}\right)$ we get

$$\frac{m}{n} = \left(\frac{1}{2k+1} + \frac{1}{4k}\right) + \left(\frac{1}{2k+2} + \frac{1}{4k-1}\right) + \ldots + \left(\frac{1}{3k} + \frac{1}{3k+1}\right) =$$

$$= \frac{p}{(2k+1)(4k)} + \frac{p}{(2k+2)(4k-1)} + \ldots + \frac{p}{(3k)(3k+1)}.$$

Because p is not divisible by any number from $\{2k+1, 2k+2, ..., 4k\}$ we get that $p|m$.

J29. Find all rational solutions of the equation

$$\left\{x^2\right\} + \{x\} = 0.99$$

Proposed by Bogdan Enescu, "B.P. Hasdeu" National College, Romania

Solution by Daniel Campos, Costa Rica.

Solution. The equation is equivalent to

$$x^2 + x - 0.99 = \lfloor x^2 \rfloor + \lfloor x \rfloor.$$

Let $x = \frac{a}{b}$, with a, b coprime integers and b greater than 0. Then, $\frac{100a^2 + 100ab - 99b^2}{100b^2}$ is an integer. This implies that

$$100|99b^2 \text{ and } b^2|100a(a+b).$$

The first one implies that $100|b^2$, while the second, since $(a, b) = 1$, implies that $b^2|100$. Then, $b = 10$.

Then, $a^2 + 10a - 99 \equiv 0 \pmod{100}$. Note that

$$a^2 + 10a - 99 \equiv a^2 + 10a - 299 \equiv (a - 13)(a + 23) \equiv 0 \pmod{100}.$$

This implies that a is odd, and that $(a - 13)(a + 23) \equiv 0 \pmod{25}$. Since $a-13 \not\equiv a+23 \pmod 5$, it follows that $a = 25k+13$ or $a = 25k+2$.

Since a is odd, it follows that it is of the form $50k + 13$ or $50k + 27$. It is easy to verify that for any rational number of the form $5k + \frac{13}{10}$ and $5k + \frac{27}{10}$, with k integer, the equality holds.

J30. Let a, b, c be three nonnegative real numbers. Prove the inequality

$$\frac{a^3+abc}{b+c}+\frac{b^3+abc}{a+c}+\frac{c^3+abc}{a+b} \geq a^2+b^2+c^2.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

First solution by Zhao Bin, HUST, China.

Solution. Without loss of generality $a \geq b \geq c$, the inequality is equivalent to:

$$\frac{a}{b+c}(a-b)(a-c)+\frac{b}{c+a}(b-a)(b-c)+\frac{c}{a+b}(c-a)(c-b) \geq 0.$$

But by $\frac{a}{b+c} \geq \frac{b}{c+a}$ and $(a-b)(a-c) \geq 0$, we have

$$\frac{a}{b+c}(a-b)(a-c)+\frac{b}{c+a}(b-a)(b-c) \geq$$

$$\geq \frac{b}{c+a}(a-b)(a-c)+\frac{b}{c+a}(b-a)(b-c) \geq \frac{b}{c+a}(a-b)^2 \geq 0.$$

Also we have

$$\frac{c}{a+b}(c-a)(c-b) \geq 0.$$

Thus we solve the problem.

Second solution by Aleksandar Ilic, Serbia.

Solution.

Rewrite the inequality in the following form:

$$\left(\frac{a^3+abc}{b+c}-a^2\right)+\left(\frac{b^3+abc}{a+c}-b^2\right)+\left(\frac{c^3+abc}{a+b}-c^2\right) \geq 0.$$

Now combine expressions in brackets to get:

$$\frac{a(a-b)(a-c)}{b+c}+\frac{b(b-a)(b-c)}{a+c}+\frac{c(c-a)(c-b)}{a+b} \geq 0.$$

When multiply both sides of equation with $(a+b)(b+c)(c+a)$ we get Schur's inequality for numbers a^2, b^2 and c^2 and $r=\frac{1}{2}$.

$$a(a^2-b^2)(a^2-c^2)+b(b^2-a^2)(b^2-c^2)+c(c^2-a^2)(c^2-b^2) \geq 0.$$

Also solved by Daniel Campos, Costa Rica; Ashay Burungale, India; Prachai K, Thailand.

Seniors

S25. Prove that in any acute-angled triangle ABC,

$$\cos^3 A + \cos^3 B + \cos^3 C + \cos A \cos B \cos C \geq \frac{1}{2}$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

First solution by Prachai K, Thailand.

Solution. Let $x = \cos A, y = \cos B, z = \cos C$. It is well known fact that

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1,$$

and therefore $x^2 + y^2 + z^2 + 2xyz = 1$.

Also from Jensen Inequality it is not difficult to find that

$$\cos A \cdot \cos B \cdot \cos C \leq \frac{1}{8}.$$

It follows that $xyz \leq \frac{1}{8}$ and $x^2 + y^2 + z^2 \geq \frac{3}{4}$.

Using the Power-Mean inequality we have

$$(x^3 + y^3 + z^3)^2 \geq \frac{1}{3}(x^2 + y^2 + z^2)^3 \geq \frac{1}{4}(x^2 + y^2 + z^2)^2,$$

or

$$2(x^3 + y^3 + z^3) \geq x^2 + y^2 + z^2.$$

Thus

$$2(x^3 + y^3 + z^3) + 2xyz \geq x^2 + y^2 + z^2 + 2xyz = 1,$$

and we are done.

Second solution by Hung Quang Tran, Hanoi National University, Vietnam.

Solution. Using the equality

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1,$$

the initial inequality becomes equivalent to

$$2(\cos^3 A + \cos^3 B + \cos^3 C) \geq \cos^2 A + \cos^2 B + \cos^2 C.$$

Using the fact that triangle ABC is acute angled we get $\cos A, \cos B, \cos C \geq 0$, and therefore

$$(1-2\cos A)^2\cos A + (1-2\cos B)^2\cos B + (1-2\cos C)^2\cos C \geq 0$$

$$4(\cos^3 A+\cos^3 B+\cos^3 C)-4(\cos^2 A+\cos^2 B+\cos^2 C)+(\cos A+\cos B+\cos C) \geq 0,$$

$$2(\cos^3 A+\cos^3 B+\cos^3 C) \geq 2(\cos^2 A+\cos^2 B+\cos^2 C)-\frac{1}{2}(\cos A+\cos B+\cos C).$$

Thus it is enough to prove

$$2(\cos^2 A+\cos^2 B+\cos^2 C)-\frac{1}{2}(\cos A+\cos B+\cos C) \geq \cos^2 A+\cos^2 B+\cos^2 C,$$

or

$$2(\cos^2 A + \cos^2 B + \cos^2 C) \geq \cos A + \cos B + \cos C.$$

Using well known inequalities

$$\cos 2A + \cos 2B + \cos 2C \geq -\frac{3}{2} \text{ and } \cos A + \cos B + \cos C \leq \frac{3}{2},$$

we have

$$(1+\cos 2A) + (1+\cos 2B) + (1+\cos 2C) \geq \frac{3}{2},$$

or

$$2(\cos^2 A + \cos^2 B + \cos^2 C) \geq \frac{3}{2} \geq \cos A + \cos B + \cos C,$$

and we are done.

Also solved by Daniel Campos, Costa Rica; Zhao Bin, HUST, China.

S26. Consider a triangle ABC and let I_a be the center of the circle that touches the side BC at A' and the extensions of sides AB and AC at C' and B', respectively. Denote by X the second intersections of the line $A'B'$ with the circle with center B and radius BA' and by K the midpoint of CX. Prove that K lies on the midline of the triangle ABC corresponding to AC.

Proposed by Liubomir Chiriac, Princeton University

First solution by David E. Narvaez, Universidad Tecnologica de Panama, Panama.

Solution. Let M be the midpoint of AC and let D be the second point of intersection of BC with the circle with center B and radius BA'. It follows, from the definition of K, that KM is parallel to XB, so it will be sufficient to show that XB is parallel to AC.

Since $\angle XBD$ is a central angle, we have that

$$\angle XBD = 2\left(\angle XA'D\right) = 2\left(\angle CA'B'\right) = 2\left(\frac{C}{2}\right) = \angle ACB,$$

which implies that XB is parallel to AC.

Second solution by Zhao Bin, HUST, China.

Solution. Denote D the midpoint of BC. Then clearly DK is the midline of the triangle BXC, corresponding to BX. Also we have

$$\angle BXA' = \angle BA'X = \angle B'A'C = \angle CB'A'.$$

Hence

$$BX \parallel B'C \parallel AC,$$

and thus it is not difficult to see that the line DK is the midline of the triangle ABC corresponding to AC,so K lines on the midline of the triangle ABC corresponding to AC. The problem is solved.

Also solved by Aleksandar Ilic, Serbia; Prachai K, Thailand.

S27. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sqrt[3]{\frac{a^2+bc}{b^2+c^2}}+\sqrt[3]{\frac{b^2+ca}{c^2+a^2}}+\sqrt[3]{\frac{c^2+ab}{a^2+b^2}} \geq \frac{9\sqrt[3]{abc}}{a+b+c}$$

Proposed by Pham Huu Duc, Australia

First solution by Ho Phu Thai, Da Nang, Vietnam.

Solution. By the AM-HM inequality:

$$\sqrt[3]{\frac{a^2+bc}{b^2+c^2}}+\sqrt[3]{\frac{b^2+ca}{c^2+a^2}}+\sqrt[3]{\frac{c^2+ab}{a^2+b^2}} \geq \frac{9}{\sqrt[3]{\frac{b^2+c^2}{a^2+bc}}+\sqrt[3]{\frac{c^2+a^2}{b^2+ca}}+\sqrt[3]{\frac{a^2+b^2}{c^2+ab}}}.$$

It suffices to prove that:

$$\frac{a+b+c}{\sqrt[3]{abc}} \geq \sqrt[3]{\frac{b^2+c^2}{a^2+bc}}+\sqrt[3]{\frac{c^2+a^2}{b^2+ca}}+\sqrt[3]{\frac{a^2+b^2}{c^2+ab}}.$$

By Holder's inequality:

$$\left(\sqrt[3]{\frac{b^2+c^2}{a^2+bc}}+\sqrt[3]{\frac{c^2+a^2}{b^2+ca}}+\sqrt[3]{\frac{a^2+b^2}{c^2+ab}}\right)^3 \leq$$

$$\leq 6(a^2+b^2+c^2)\left(\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}\right).$$

We are now to show that:

$$\frac{(a+b+c)^3}{abc} \geq 6(a^2+b^2+c^2)\left(\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}\right)$$

$$\Leftrightarrow \frac{(a+b+c)^3}{abc}-27 \geq 3\sum_{cyc}\left(\frac{2a^2+2b^2+2c^2}{c^2+ab}-3\right)$$

$$\Leftrightarrow \frac{\frac{1}{2}(a+b+c)\sum_{cyc}(b-c)^2+3\sum_{cyc}a(b-c)^2}{abc} \geq$$

$$\geq 3\sum_{cyc}\frac{3(b-c)^2}{2(a^2+bc)}+3\sum_{cyc}(b-c)^2\frac{(b+c)(b+c-a)}{2(b^2+ca)(c^2+ab)}$$

$$\Leftrightarrow \sum_{cyc}(b-c)^2\left(\frac{7a+b+c}{abc}-\frac{9}{a^2+bc}-\frac{3(b+c)(b+c-a)}{(b^2+ca)(c^2+ab)}\right) \geq 0.$$

Consider the expressions S_a, S_b, S_c before $(b-c)^2, (c-a)^2, (a-b)^2$, respectively. We will point $S_a, S_b, S_c \geq 0$ out.

$$S_a = \frac{7a+b+c}{abc} - \frac{9}{a^2+bc} - \frac{3(b+c)(b+c-a)}{(b^2+ca)(c^2+ab)} \geq 0$$

$$\Leftrightarrow 7a^4b^3 + 7a^4c^3 + 7a^5bc + ab^5c + abc^5 + a^3b^4 + a^3c^4 + b^4c^3 + b^3c^4 +$$

$$3a^3b^2c^2 + 3a^2b^3c^2 + 3a^2b^2c^3 + 2a^4b^2c + 2a^4bc^2 - 4ab^3c^3 - 2a^2b^4c - 2a^2bc^4 \geq 0.$$

This is obviously true, by AM-GM:

$$b^4c^3 + b^3c^4 + a^2b^3c^2 + a^2b^2c^3 \geq 4ab^3c^3,$$

$$a^3b^4 + ab^5c + a^2b^3c^2 \geq 3a^2b^4c,$$

$$a^3c^4 + abc^5 + a^2b^2c^3 \geq 3a^2bc^4.$$

Similarly, $S_b, S_c \geq 0$ for any numbers $a, b, c > 0$.
Our proof is complete. Equality occurs if and only if $a = b = c$.

Second solution by Zhao Bin, HUST, China.

Solution. If one of a, b, c is zero, then clearly the inequality is true. We may assume $a, b, c > 0$.

By AM-GM inequality we have:

$$\sqrt[3]{abc}\sqrt[3]{a^2+bc}\sqrt[3]{a^2+bc}\sqrt[3]{b^2+c^2} = \sqrt[3]{b(a^2+bc)}\sqrt[3]{c(a^2+bc)}\sqrt[3]{a(b^2+c^2)}$$

$$\leq \frac{a^2b + b^2a + b^2c + c^2b + a^2c + c^2a}{3}$$

Thus:

$$\sqrt[3]{\frac{a^2+bc}{abc(b^2+c^2)}} = \frac{a^2+bc}{\sqrt[3]{abc}\sqrt[3]{a^2+bc}\sqrt[3]{a^2+bc}\sqrt[3]{b^2+c^2}} \geq$$

$$\frac{3(a^2+bc)}{a^2b + b^2a + b^2c + c^2b + a^2c + c^2a}.$$

Analogously,

$$\sqrt[3]{\frac{b^2+ca}{abc(c^2+a^2)}} \geq \frac{3(b^2+ca)}{a^2b + b^2a + b^2c + c^2b + a^2c + c^2a}$$

and

$$\sqrt[3]{\frac{c^2+ab}{abc(a^2+b^2)}} \geq \frac{3(c^2+ab)}{a^2b+b^2a+b^2c+c^2b+a^2c+c^2a}$$

Adding three inequalities above, we get:

$$\sqrt[3]{\frac{a^2+bc}{b^2+c^2}}+\sqrt[3]{\frac{b^2+ca}{c^2+a^2}}+\sqrt[3]{\frac{c^2+ab}{a^2+b^2}} \geq \frac{3\sqrt[3]{abc}(a^2+b^2+c^2+ab+bc+ca)}{a^2b+b^2a+b^2c+c^2b+a^2c+c^2a}.$$

Thus to prove the original inequality, it suffices to prove

$$\frac{a^2+b^2+c^2+ab+bc+ca}{a^2b+b^2a+b^2c+c^2b+a^2c+c^2a} \geq \frac{3}{a+b+c}.$$

But this is equivalent to

$$a^3+b^3+c^3+3abc \geq a^2b+b^2a+b^2c+c^2b+a^2c+c^2a,$$

which is the Schur's Inequality, and the problem is solved.

S28. Let M be a point in the plane of triangle ABC. Find the minimum of

$$MA^3 + MB^3 + MC^3 - \frac{3}{2}R \cdot MH^2,$$

where H is the orthocenter and R is the circumradius of the triangle ABC.

Proposed by Hung Quang Tran, Hanoi, Vietnam

Solution by Hung Quang Tran, Hanoi, Vietnam.

Solution. Using AM-GM inequality we have

$$\frac{MA^3}{R} + \frac{R^2 + MA^2}{2} \geq \frac{MA^3}{R} + R \cdot MA \geq 2MA^2,$$

or

$$\frac{MA^3}{R} \geq \frac{3}{2}MA^2 - \frac{R^2}{2}.$$

Analogously

$$\frac{MB^3}{R} \geq \frac{3}{2}MB^2 - \frac{R^2}{2}, \quad \frac{MC^3}{R} \geq \frac{3}{2}MC^2 - \frac{R^2}{2}.$$

Thus

$$\frac{MA^3 + MB^3 + MC^3}{R} \geq \frac{3}{2}(MA^2 + MB^2 + MC^2) - \frac{3}{2}R^2.$$

$$MA^2 + MB^2 + MC^2 = (\vec{MO} + \vec{OA})^2 + (\vec{MO} + \vec{OB})^2 + (\vec{MO} + \vec{OC})^2 =$$
$$3MO^2 + 2\vec{MO}(\vec{OA} + \vec{OB} + \vec{OC}) + 3R^2 = MO^2 + 2\vec{MO} \cdot \vec{OH} =$$
$$= 3MO^2 - (OM^2 + OH^2 - MH^2) + 3R^2 \geq 3R^2 - OH^2 + MH^2.$$

Hence

$$\frac{MA^3 + MB^3 + MC^3}{R} \geq \frac{3}{2}(3R^2 - OH^2 + MH^2) - \frac{3}{2}R^2,$$

and therefore

$$MA^3 + MB^3 + MC^3 - \frac{3}{2}R \cdot MH^2 \geq 3R^2 - \frac{3}{2}R \cdot OH^2 = \text{const}.$$

Clearly the equality holds when $M \equiv O$.

S29. Prove that for any real numbers a, b, c the following inequality holds

$$3(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ac + a^2) \geq a^3b^3 + b^3c^3 + c^3a^3.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

First solution by Zhao Bin, HUST, China.

Solution. Clearly it is enough to consider the case when $a, b, c \geq 0$. We have

$$(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2) = \sum_{sym} a^4b^2 - \sum_{cyc} a^3b^3 - \sum_{cyc} a^4bc + a^2b^2c^2.$$

The inequality is equivalent to

$$3\sum_{sym} a^4b^2 - 3\sum_{cyc} a^3b^3 - 3\sum_{cyc} a^4bc + 3a^2b^2c^2 \geq 0,$$

which is also equivalent to

$$\sum_{cyc} \left(2c^4 + 3a^2b^2 - abc(a+b+c)\right)(a-b)^2 \geq 0.$$

Without loss of generality suppose $a \geq b \geq c$, and let

$$S_a = 2a^4 + 3b^2c^2 - abc(a+b+c),$$

$$S_b = 2b^4 + 3c^2a^2 - abc(a+b+c),$$

$$S_c = 2c^4 + 3a^2b^2 - abc(a+b+c).$$

We have

$$S_a = 2a^4 + 3b^2c^2 - abc(a+b+c) \geq a^4 + 2a^2bc - abc(a+b+c) \geq 0,$$

$$S_c = 2c^4 + 3a^2b^2 - abc(a+b+c) \geq 3a^2b^2 - abc(a+b+c) \geq 0,$$

also we have

$$S_a + 2S_b = 2a^4 + 3b^2c^2 + 4b^4 + 6c^2a^2 - 3abc(a+b+c) \geq$$

$$a^4 + 2a^2bc + 8b^2ca - 3abc(a+b+c) \geq 0,$$

$$S_c + 2S_b = 2c^4 + 3a^2b^2 + 4b^4 + 6c^2a^2 - 3abc(a+b+c) \geq$$

$$(3a^2b^2 + 3a^2c^2) + 3a^2c^2 - 3abc(a+b+c) \geq 0.$$

Then if $S_b \geq 0$ the last inequality (1) is true. If $S_b < 0$ then

$$\sum_{cyc} S_a(b-c)^2 \geq S_a(b-c)^2 + 2S_b(b-c)^2 + 2S_b(a-b)^2 + S_c(a-b)^2 \geq 0.$$

The inequality (1) is also true and the inequality is solved.

Second solution by Daniel Campos, Costa Rica.

Solution. Note that $x^2 - xy + y^2 \geq |x|^2 - |x||y| + |y|^2 \geq 0$ and that $|x|^3|y|^3 \geq x^3y^3$, then it is enough to prove it for a, b, c nonnegative reals.

Recall the identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)((x-y)^2 + (y-z)^2 + (z-x)^2),$$

then the inequality is equivalent to

$$\begin{aligned} 3\prod_{cyc}((a-b)^2 + ab) - 3a^2b^2c^2 &\geq a^3b^3 + b^3c^3 + c^3a^3 - 3a^2b^2c^2 \\ &= \frac{1}{2}(ab+bc+ca)\sum_{cyc} c^2(a-b)^2. \end{aligned}$$

Then, we have to prove that

$$6\prod_{cyc}((a-b)^2 + ab) - 6a^2b^2c^2 - (ab+bc+ca)\sum_{cyc} c^2(a-b)^2 \geq 0,$$

or that

$$\sum_{cyc}(a-b)^2(2(a-c)^2(b-c)^2 + 3c(a(b-c)^2 + b(a-c)^2) + 6abc^2 - c^2(ab+bc+ca)) \tag{1}$$

is greater or equal than 0.

After expanding we have that

$$2(a-c)^2(b-c)^2 + 3c(a(b-c)^2 + b(a-c)^2) + 6abc^2 - c^2(ab+bc+ca)$$

equals to

$$2c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 + abc^2 - a^2bc - ab^2c - 2ac^3 - 2bc^3,$$

or

$$(c^4 + a^2c^2 - 2ac^3) + (c^4 + b^2c^2 - 2bc^3) + (a^2b^2 + a^2c^2 - 2a^2bc)$$

$$+(a^2b^2+b^2c^2-2ab^2c)+a^2bc+ab^2c+abc^2.$$

In the last expression, by AM-GM, each term inside the parenthesis is nonnegative, which implies (1) is a sum of nonnegative terms and this completes the proof.

Third solution by Aleksandar Ilic, Serbia.

Solution. When we multiply both sides with $(a+b)(a+c)(b+c)$ we get:

$$3(a^3+b^3)(a^3+c^3)(b^3+c^3) \geq (a^3b^3+a^3c^3+b^3c^3)(a+b)(a+c)(b+c).$$

Now we get free of brackets and gather similar terms. Using symmetrical sums, we can rewrite inequality in following form:

$$3\sum_{sym} a^6b^3 + \sum_{sym} a^3b^3c^3 \geq \sum_{sym} a^4b^4c + \sum_{sym} a^5b^4 + \sum_{sym} a^5b^3c + \sum_{sym} a^4b^3c^2.$$

We use Schur's inequality:

$$\sum_{sym} x^3 + \sum_{sym} xyz \geq 2\sum_{sym} x^2y.$$

For numbers $x=a^2b$, $y=b^2c$ and $z=c^2a$ we get:

$$\sum_{sym} a^6b^3 + \sum_{sym} a^3b^3c^3 \geq \sum_{sym} a^4b^4c + \sum_{sym} a^5b^2c^2.$$

Because $[5,2,2] \succ [4,3,2]$ from Miurhead's inequality we get

$$\sum_{sym} a^5b^2c^2 \geq \sum_{sym} a^4b^3c^2.$$

Finally, we substitute last inequality in the one before last and add two inequalities with symmetrical sums.

$$\sum_{sym} a^6b^3 + \sum_{sym} a^3b^3c^3 \geq \sum_{sym} a^4b^4c + \sum_{sym} a^4b^3c^2.$$

$$\sum_{sym} a^6b^3 \geq \sum_{sym} a^5b^4.$$

$$\sum_{sym} a^6b^3 \geq \sum_{sym} a^5b^3c.$$

Fourth solution by Dr. Titu Andreescu, University of Texas at Dallas.

Solution. Let us prove the following lemma:

Lemma. For any real numbers x, y we have

$$3(x^2 - xy + y^2)^3 \geq x^6 + x^3y^3 + y^6.$$

Denote $s = x + y$ and $p = xy$. Then clearly $s^2 - 4p \geq 0$ and we have

$$3(x^2 - xy + y^2)^3 = 3(s^2 - 3p)^3 = 3((s^2 - 2p) - p)^3 =$$

$$= 3(s^2 - 2p)^3 - 9(s^2 - 2p)^2 p + 9(s^2 - 2p)p^2 - 3p^3,$$

and

$$x^6 + x^3y^3 + y^6 = (x^2 + y^2)((x^2 + y^2)^2 - 3x^2y^2) + x^3y^3 =$$

$$= (s^2 - 2p)((s^2 - 2p)^2 - 3p^2) + p^3 = (s^2 - 2p)^3 - 3(s^2 - 2p)p^2 + p^3.$$

Thus it is enough to prove that

$$2(s^2 - 2p)^3 - 9(s^2 - 2p)^2 p + 12(s^2 - 2p)p^2 - 4p^3 \geq 0,$$

or

$$2(s^2 - 2p)^2(s^2 - 4p) - 5(s^2 - 2p)^2 p(s^2 - 4p) + 2p(s^2 - 4p) \geq 0.$$

Last inequality is equivalent to

$$(s^2 - 4p)(2(s^2 - 2p)^2 - 5(s^2 - 2p)^2 p + 2p) \geq 0,$$

or

$$(s^2 - 4p)(2(s^2 - 2p)(s^2 - 4p) - p(s^2 - 4p)) \geq 0.$$

That is $(s^2 - 4p)^2(2s^2 - 5p) \geq 0$ and lemma is proven.

Returning back to the problem and using our lemma we have

$$3(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ac + a^2) \geq$$

$$\geq (a^6 + a^3b^3 + b^6)^{\frac{1}{3}}(b^6 + b^3c^3 + c^6)^{\frac{1}{3}}(c^6 + c^3a^3 + a^6)^{\frac{1}{3}} \geq a^3b^3 + b^3c^3 + c^3a^3.$$

Last inequality is due Holder, combining triples

$$(a^3b^3, b^6, a^6),\ (b^6, b^3c^3, c^6),\ (a^6, c^6, a^3c^3).$$

S30. Let $p > 5$ be a prime number and let

$$S(m) = \sum_{i=0}^{\frac{p-1}{2}} \frac{m^{2i}}{2i}.$$

Prove that the numerator of $S(1)$ is divisible by p if and only if the numerator of $S(3)$ is divisible by p.

Proposed by Iurie Boreico, Moldova

Solution by Iurie Boreico, Moldova

Solution. We shall consider congruence in rational numbers.
Let $\frac{a}{b}$ in lowest terms be divisible by p if p divides a.
Now we have to prove that $p|S(1)$ if and only if $p|S(3)$.
Let $0 < k < p$. Then $\frac{\binom{p}{k}}{p} = \frac{(p-1)!}{k!(p-k)!}$, we have

$$(p-k)! \equiv (-1)^{p-k}(p-1)(p-2)\dots k.$$

Therefore we conclude

$$\frac{\binom{p}{k}}{p} \equiv (-1)^{k-1}\frac{1}{k}(\text{mod } p).$$

Consider the sum $Q(m) = (m+1)^p - (m-1)^p - 2$. It is clear from Newton's Binomial Theorem and the result above that

$$S(m) \equiv \frac{1}{-2p}Q(m)(\text{mod } p),$$

because

$$Q(m) = 2p(m^{p-1} + \frac{\binom{p}{3}}{p}m^{p-3} + ... + \frac{\binom{p}{p-2}}{p}m^2) \equiv$$

$$\equiv 2p\left(m^{p-1} + (-1)^{3-1}\frac{m^{p-3}}{3} + ... + (-1)^{p-2-1}\frac{m^2}{p-2}\right) \equiv$$

$$\equiv -2p\left(\frac{m^{p-1}}{p-1} + \frac{m^{p-3}}{p-3} + ... + \frac{m^2}{2}\right)(\text{mod } p).$$

Hence $p|S(m)$ if an only if $p^2|Q(m)$ (for $0 < m < p$).
Therefore we must prove that $p^2|Q(1)$ if and only if $p^2|Q(3)$.
But $Q(1) = 2^p - 2$ and $Q(3) = 4^p - 2^p - 2 = (2^p - 2)(2^p + 1)$. As $2^p + 1$ is not divisible by p, the conclusion follows.

Undergraduate

U25. Calculate the following sum $\displaystyle\sum_{k=0}^{\infty} \frac{2k+1}{(4k+1)(4k+3)(4k+5)}$.

Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania

First solution by Vishal Lama, Southern Utah University

Solution. Let $S = \sum_{k=0}^{\infty} \frac{2k+1}{(4k+1)(4k+3)(4k+5)}$.
Using partial fractions, we note that

$$a_k = \frac{2k+1}{(4k+1)(4k+3)(4k+5)} = \frac{1}{16} \cdot \frac{1}{4k+1} + \frac{2}{16} \cdot \frac{1}{4k+3} - \frac{3}{16} \cdot \frac{1}{4k+5}.$$

Let $S_n = \sum_{k=0}^{n} a_k$. Then,

$$S_n = \sum_{k=0}^{n} \left(\frac{1}{16} \cdot \frac{1}{4k+1} + \frac{2}{16} \cdot \frac{1}{4k+3} - \frac{3}{16} \cdot \frac{1}{4k+5} \right) =$$

$$= \frac{1}{16} \sum_{k=0}^{n} \left(\frac{1}{4k+1} - \frac{1}{4k+5} \right) + \frac{2}{16} \sum_{k=0}^{n} \left(\frac{1}{4k+3} - \frac{1}{4k+5} \right) =$$

$$= \frac{1}{16} \left(1 - \frac{1}{4n+5} \right) + \frac{2}{16} \sum_{k=0}^{n} \left(\frac{1}{4k+3} - \frac{1}{4k+5} \right).$$

Thus, $S = \lim_{n\to\infty} S_n$

$$S = \frac{1}{16} + \frac{2}{16} \sum_{k=0}^{\infty} \left(\frac{1}{4k+3} - \frac{1}{4k+5} \right)$$

$$\Rightarrow S = \frac{1}{16} + \frac{2}{16} \left(\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \dots \right).$$

But, then we have

$$\int_0^1 \frac{dt}{1+t^2} = \tan^{-1} t \Big|_0^1 = \frac{\pi}{4}, \text{ (where } |t| < 1)$$

$$\Rightarrow \frac{\pi}{4} = \int_0^1 (1 - t^2 + t^4 - t^6 + t^8 - \dots)\, dt$$

$$\Rightarrow \frac{\pi}{4} = (t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \dots)\Big|_0^1$$

$$\Rightarrow \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \dots = 1 - \frac{\pi}{4}.$$

Using the above result we get

$$S = \frac{1}{16} + \frac{2}{16}\left(1 - \frac{\pi}{4}\right) = \frac{6-\pi}{32}.$$

Second solution by Aleksandar Ilic, Serbia.

Solution. We have to divide series into some sums with nicer form. The following identity can be interesting.

$$\frac{2k+1}{(4k+1)(4k+3)(4k+5)} = \frac{1}{16} \cdot \frac{1}{4k+1} + \frac{1}{8} \cdot \frac{1}{4k+3} - \frac{3}{16} \cdot \frac{1}{4k+5}.$$

We get this the same way we disunite rational functions and verification is strait-forward. First and third sum are the same, except the first term, so summing from $k = 0$ to infinity we have:

$$S = \frac{1}{16} \cdot \sum_{k=0}^{\infty} \frac{1}{4k+1} + \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{4k+3} - \frac{3}{16} \sum_{k=0}^{\infty} \frac{1}{4k+5}.$$

Rearranging and grouping terms, we get:

$$S = \frac{3}{16} + \left(\frac{1}{16} - \frac{3}{16}\right) \sum_{k=0}^{\infty} \frac{1}{4k+1} + \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{4k+3} =$$

$$= \frac{3}{16} - \frac{1}{8} \sum_{k=0}^{\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+3}\right) =$$

$$= \frac{3}{16} - \frac{1}{8}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) = \frac{3}{16} - \frac{1}{8} \cdot \frac{\pi}{4}.$$

Using well-known summation for number π, the series equals $\frac{6-\pi}{32} \approx 0.089325$.

Also solved by Ashay Burungale, India; Jean-Charles Mathieux, Dakar University, Sénégal.

U26. Let $f : [a, b] \to \mathbb{R}$ ($0 < a < b$) be a continuous function on $[a, b]$ and differentiable on (a, b). Prove that there is a $c \in (a, b)$ such that

$$\frac{2}{a-c} < \ f'(c) < \frac{2}{b-c}$$

Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania

First solution by Bin Zhao, HUST, China.

Solution. If there is a $x_1, x_2 \in (a, b)$ such $f'(x_1) \geq 0, f'(x_2) \leq 0$, then by Darboux's Theorem we have there is a c between x_1, x_2, such that $f'(c) = 0$, then c will satisfy the condition.

If not we may assume $f'(x) > 0, x \in (a, b)$ (because the proof will be similar for $f'(x) < 0, x \in (a, b)$). Then assume the contrary, which means there is not a $c \in (a, b)$ such that

$$\frac{2}{a-c} < f'(c) < \frac{2}{b-c}.$$

It follows that we have $f'(x) \geq \frac{2}{b-c}$.

Let $x_k = b - \frac{1}{2^k}(b-a), k = 1, 2, \ldots$. Then

$$f(x_1) - f(a) = f\left(\frac{a+b}{2}\right) - f(a) = f'(\xi_1)\frac{b-a}{2} \geq \frac{2}{b-\xi_1} \cdot \frac{b-a}{2} \geq 1,$$

and

$$f(x_{k+1}) - f(x_k) = f'(\xi_{k+1})(x_{k+1} - x_k) \geq \frac{2}{b-\xi_{k+1}} \cdot \frac{b-a}{2^{k+1}} \geq 1,$$

$k = 1, 2, \ldots$, and $\xi_1 \in (a, x_1), x_{k+1} \in (x_k, x_k + 1)$.

We have $f(x_n) - f(a) \geq n$, which will be in contradiction with $f(x_n) - f(a) \leq 2M(M = \max_{a \leq x \leq b} f(x))$, when n is large enough. The problem is solved.

Second solution by Aleksandar Ilic, Serbia.

Solution. Notice that $\frac{1}{a-c}$ is less than zero, and number $\frac{1}{b-c}$ is greater than zero. If there exist $c \in (a, b)$ such that $f'(c) = 0$, problem is solved. From Darboux's theorem function $f'(x)$ always has the same sign. Let $f'(x) > 0$ for every $x \in (a, b)$. Now we proceed by contradiction: assume that for every $c \in (a, b)$ we have

$$f'(c) \geq \frac{2}{b-c}.$$

We can integrate inequality in interval (a, x), and get

$$f(x) - f(a) = \int_a^x f'(c)dc \geq \int_a^x \frac{2dc}{b-c} = 2\left(\ln(b-a) - \ln(b-x)\right).$$

If we let $x \to b$, left side becomes $f(b) - f(a)$ and right side is

$$2\ln(b-a) - \lim_{x\to b} \ln(x-b) \to +\infty.$$

This is impossible, since left side is always greater of equal then right side. Contradiction! Case $f'(x) < 0$ can be considered in similar manner.

Third solution by José Luis Díaz-Barrero, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.

Solution. Consider the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = (x-a)(x-b)\exp[\,f(x)]$$

Since F is continuous function on $[a, b]$, derivable in (a, b) and $F(a) = F(b) = 0$, then by Rolle's theorem there exists $c \in (a, b)$ such that $F'(c) = 0$. We have

$$F'(x) = [x - b + x - a + (x-a)(x-b)\ f'(x)]\exp[\,f(x)]\,,$$

and

$$2c - a - b + (c-a)(c-b)\ f'(c) = 0.$$

From the preceding and from ($0 < a < b$) immediately follows

$$\frac{2}{a-c} < \ f'(c) = \frac{a+b-2c}{(a-c)(b-c)} < \frac{2}{b-c}.$$

In fact, since $a - c < 0$, then

$$\frac{2}{a-c} < \frac{a+b-2c}{(a-c)(b-c)} \Leftrightarrow 2 > \frac{a+b-2c}{b-c} \Leftrightarrow 2b-2c > a+b-2c \Leftrightarrow b > a,$$

and

$$\frac{a+b-2c}{(a-c)(b-c)} < \frac{2}{b-c} \Leftrightarrow \frac{a+b-2c}{a-c} < 2 \Leftrightarrow a+b-2c > 2a-2c \Leftrightarrow b > a.$$

This completes the proof.

U27. Let k be a positive integer. Evaluate

$$\int_0^1 \left\{\frac{k}{x}\right\}^2 dx$$

where $\{a\}$ is the *fractional part* of a.

Proposed by Ovidiu Furdui, Western Michigan University

Solution by Ovidiu Furdui, Western Michigan University.

Solution. The integral equals

$$k\left(\ln(2\pi) - \gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + 2k\ln k - 2k - 2\ln(k!)\right),$$

where $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$ is the *Euler-Mascheroni* constant. If we make the substitution $\frac{k}{x} = t$, we get that

$$I = \int_0^1 \left\{\frac{k}{x}\right\}^2 dx = k\int_k^\infty \frac{\{t\}^2}{t^2}dt = k\sum_{l=k}^\infty \int_l^{l+1} \frac{(t-l)^2}{t^2}dt =$$

$$k\sum_{l=k}^\infty \int_l^{l+1}\left(1 - \frac{2l}{t} + \frac{l^2}{t^2}\right)dt = k\sum_{l=k}^\infty\left(1 - 2l\ln\frac{l+1}{l} + \frac{l}{l+1}\right) =$$

$$= k\sum_{l=k}^\infty\left(2 - 2l\ln\frac{l+1}{l} - \frac{1}{l+1}\right).$$

Let S_n be the n^{th} partial sum of the preceding series, i.e.,

$$S_n = \sum_{l=k}^n\left(2 - 2l\ln\frac{l+1}{l} - \frac{1}{l+1}\right).$$

This series is a telescoping series, so we obtain

$$S_n = 2(n-k+1) - \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{1+n}\right) - 2\sum_{l=k}^n l\ln\frac{l+1}{l} =$$

$$= 2(n-k+1) - \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{1+n}\right) -$$

$$-2\left[n\ln(n+1) - k\ln k - \ln\frac{n!}{k!}\right] =$$

$$= 2(n-k+1) - \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{1+n}\right) -$$

$$-2n\ln(n+1) + 2k\ln k + 2\ln(n!) - 2\ln(k!).\ (1)$$

For calculating $\lim\limits_{n\to\infty} S_n$, we will make use of *Stirling's formula*, i.e.,

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

It follows that

$$2\ln n! \approx \ln(2\pi) + (2n+1)\ln n - 2n.\ (2)$$

Combining (1) and (2), we get after straightforward calculations that

$$S_n = 2(1-k) + \ln(2\pi) + 2k\ln k - 2\ln(k!) - 2n\ln\frac{n+1}{n} -$$

$$-\left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{1+n} - \ln n\right) \to$$

$$\to -2k + \ln(2\pi) + 2k\ln k - 2\ln k! - \left(\gamma - 1 - \frac{1}{2} - \cdots - \frac{1}{k}\right)$$

$$= \ln(2\pi) - \gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + 2k\ln k - 2k - 2\ln(k!).$$

Thus,

$$\int_0^1 \left\{\frac{k}{x}\right\}^2 dx = k\left(\ln(2\pi) - \gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + 2k\ln k - 2k - 2\ln(k!)\right).$$

Remark. When $k = 1$ the following integral formulae holds.

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 dx = \ln 2\pi - \gamma - 1.44$$

U28. Let f be the function defined by

$$f(x) = \sum_{n\geq 1} |\sin n| \cdot \frac{x^n}{1-x^n}.$$

Find in a closed form a function g such that $\lim_{x\to 1^-} \frac{f(x)}{g(x)} = 1$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

No solutions received.

U29. Let A be a square matrix of order n, for which there is a positive integer k such that $kA^{k+1} = (k+1)A^k$. Prove that $A - I_n$ is invertible and find its inverse.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

First solution by Bin Zhao, HUST, China.

Solution. Let $B = A - I_n$,then we have:

$$k(B + I_n)^{k+1} = (k+1)(B + I_n)^k$$

which is equivalent to

$$k\left(\sum_{i=0}^{k+1}\binom{k+1}{i}B^i\right) = (k+1)\left(\sum_{i=0}^{k}\binom{k}{i}B^i\right)$$

$$\iff \sum_{i=1}^{k+1}\left(k\binom{k+1}{i} - (k+1)\binom{k}{i}\right)B^i = I_n$$

$$\iff B\left(\sum_{i=0}^{k}\left(k\binom{k+1}{i+1} - (k+1)\binom{k}{i+1}\right)B^i\right) = I_n.$$

Thus we have $A - I_n$ is invertible, and its inverse is

$$\sum_{i=0}^{k}\left(k\binom{k+1}{i+1} - (k+1)\binom{k}{i+1}\right)B^i,$$

where $B = A - I_n$.

Second solution by Jean-Charles Mathieux, Dakar University, Sénégal.

Solution. You can show that $A - I_n$ is invertible without exhibiting its inverse. For instance, suppose that $A - I_n$ is not invertible, then there is a non zero vector X such that $AX = X$, since $kA^{k+1} = (k+1)A^k$, you have $kX = (k+1)X$ which is a contradiction.

However we can use another approach:
$kA^k(A - I_n) - (A^k - I_n) = kA^{k+1} - (k+1)A^k + I_n = I_n$,
and $A_k - I_n = (A - I_n)\sum_{i=0}^{k-1} A^i$.

So $(A - I_n)(kA^k - A^{k-1} - A^{k-2} - \cdots - I_n) = I_n$, which shows that $(A - I_n)$ is invertible and that $(A - I_n)^{-1} = (kA^k - A^{k-1} - A^{k-2} - \cdots - I_n)$.

U30. Let n be a positive integer. What is the largest cardinal of a finite subgroup G of $GL_n(\mathbb{Z})$ such that for any matrix $A \in G$, all elements of $A - I_n$ are even?

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Solution by Jean-Charles Mathieux, Dakar University, Sénégal.

Solution. Let us present a sketch of the proof. Let $m = |G|$. If $A \in G$, $A^m = I_n$ so A is diagonalisable, in $\mathcal{M}_n(\mathbb{C})$ and its eigenvalues λ are such that $|\lambda| \leqslant 1$.

There exist $B \in \mathcal{M}_n(\mathbb{Z})$ such that $A = I_n + 2B$. B is also diagonalisable, in $\mathcal{M}_n(\mathbb{C})$ and its eigenvalues μ are such that $|\mu| \leqslant 1$. In fact, since $\mu = \frac{\lambda - 1}{2}$, $|\mu| = 1$ iff $\lambda = -1$. Then you show that only 0 and 1 could be eigenvalues of B.

Reciprocally, we check that $G = \{\operatorname{diag}(\pm 1, \ldots, \pm 1)\}$ satisfies the assumptions.

So the largest cardinal of a finite subgroup G of $\mathrm{GL}_n(\mathbb{Z})$ such that for any matrix $A \in G$, all elements of $A - I_n$ are even is 2^n.

Olympiad

O25. For any triangle ABC, prove that

$$\cos\frac{A}{2}\cot\frac{A}{2}+\cos\frac{B}{2}\cot\frac{B}{2}+\cos\frac{C}{2}\cot\frac{C}{2}\geq\frac{\sqrt{3}}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)$$

Proposed by Darij Grinberg, Germany

First solution by Zhao Bin, HUST, China.

Solution. Denote a, b, c be the three side of the triangle, and

$$a = y + z, b = z + x, c = x + y.$$

We have:

$$r = \sqrt{\frac{xyz}{x+y+z}}$$

$$\cos\frac{A}{2} = \frac{x}{\sqrt{x^2+r^2}}, \cos\frac{B}{2} = \frac{y}{\sqrt{y^2+r^2}}, \cos\frac{C}{2} = \frac{z}{\sqrt{z^2+r^2}},$$

and

$$\cos\frac{A}{2} = \frac{x}{r}, \cos\frac{B}{2} = \frac{y}{r}, \cos\frac{C}{2} = \frac{z}{r}$$

Then the inequality is equivalent to:

$$\frac{x^2}{\sqrt{4x(x+y+z)\cdot 3(x+y)(x+z)}} + \frac{y^2}{\sqrt{4y(x+y+z)\cdot 3(y+x)(y+z)}}$$

$$+\frac{z^2}{\sqrt{4z(x+y+z)\cdot 3(z+x)(z+y)}} \geq \frac{1}{4}.$$

But we have:

$$2\sqrt{4x(x+y+z)\cdot 3(x+y)(x+z)} \leq 4x(x+y+z) + 3(x+y)(x+z) =$$

$$= 7x(x+y+z) + 3yz,$$

$$2\sqrt{4y(x+y+z)\cdot 3(y+x)(y+z)} \leq 4y(x+y+z) + 3(y+x)(y+z) =$$

$$= 7y(x+y+z) + 3zx,$$

$$2\sqrt{4z(x+y+z)\cdot 3(z+x)(z+y)} \leq 4z(x+y+z) + 3(z+x)(z+y) =$$

$$= 7z(x+y+z) + 3xy.$$

Thus it suffices to prove:

$$\frac{x^2}{7x(x+y+z)+3yz}+\frac{y^2}{7y(x+y+z)+3zx}+\frac{z^2}{7z(x+y+z)+3xy}\geq\frac{1}{8}.$$

But by Cauchy Inequality we have:

$$\frac{x^2}{7x(x+y+z)+3yz}+\frac{y^2}{7y(x+y+z)+3zx}+\frac{z^2}{7z(x+y+z)+3xy}$$

$$\geq\frac{(x+y+z)^2}{7(x+y+z)^2+3(xy+yz+zx)}\geq\frac{1}{8}.$$

So we solved the inequality.

Second solution by David E. Narvaez, Universidad Tecnologica, Panama.

Solution. From Jensen's inequality we have that

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\geq\sqrt{3}.$$

and

$$\sin\frac{A}{2}\sin\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}\geq\frac{3}{4}.$$

thus

$$\frac{2}{3}\left(\sum_{cyc}\tan\frac{A}{2}\right)\left(\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}\right)\geq\frac{\sqrt{3}}{2}.$$

Let us assume, without loss of generality, that $A\geq B\geq C$. Then $\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\geq\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ and $\sin\frac{A}{2}\sin\frac{B}{2}\geq\sin\frac{C}{2}\sin\frac{A}{2}\geq\sin\frac{B}{2}\sin\frac{C}{2}$ and by Chebychev's inequality we get

$$\sum_{cyc}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\sin\frac{B}{2}\sin\frac{C}{2}\geq$$

$$\geq\frac{1}{3}\left(\sum_{cyc}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\right)\left(\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}\right)\geq\frac{\sqrt{3}}{2},$$

but

$$\begin{aligned}
\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\sin\frac{B}{2}\sin\frac{C}{2} &= \left(\frac{\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{B}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)\sin\frac{B}{2}\sin\frac{C}{2},\\
&= \sin\frac{B+C}{2}\tan\frac{B}{2}\tan\frac{C}{2},\\
\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\sin\frac{B}{2}\sin\frac{C}{2} &= \cos\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}.
\end{aligned}$$

and replacing this and similar identities for every term in the left hand side of our last inequality we have

$$\sum_{cyc}\cos\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\geq\frac{\sqrt{3}}{2}.$$

Multiplying this inequality by $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$ we get

$$\cos\frac{A}{2}\cot\frac{A}{2}+\cos\frac{B}{2}\cot\frac{B}{2}+\cos\frac{C}{2}\cot\frac{C}{2}\geq\frac{\sqrt{3}}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right),$$

and we are done.

O26. Consider a triangle ABC and let O be its circumcenter. Denote by D the foot of the altitude from A and by E the intersection of AO and BC. Suppose tangents to the circumcircle of triangle ABC at B and C intersect at T and that AT intersects this circumcircle at F. Prove that the circumcircles of triangles DEF and ABC are tangent.

Proposed by Ivan Borsenco, University of Texas at Dallas

Solution by David E. Narvaez, Universidad Tecnologica de Panama, Panama.

Solution. Let ω, ω' and ω'' be the circumcircles of triangles ABC, TDE and ADE, respectively; let X and F' be the points where the line BC cuts the tangent to ω through A and the line AT. It is a well known fact that AT is the symmedian corresponding to the vertex A in triangle ABC*, and since points X, B, F' and C are harmonic conjugates, F' is in the polar line of X, and so is A, so AT' is the polar line of X, which implies that the tangent to ω through F passes through X.

We claim that XA is tangent to ω'', and from the power of the point X with respect to ω'' we get that

$$XA^2 = XD \cdot XE,$$

which happens to show that the powers of the point X with respect to ω and ω' are equal. Thus X is in the radical axis of ω and ω'. Since F is a point of intersection of these circumferences and the radical axis XF is tangent to ω', it is a tangent to ω too, and it follow that this two circumferences are tangent, as we wished to show.

To prove our claim, consider that $m\angle XAB = m\angle ACB$, because XA is tangent to ω; and $m\angle BAD = m\angle EAC$, because the orthocenter and the circumcenter are isogonal conjugates. Then

$$m\angle XAD = m\angle XAB + m\angle BAD = m\angle ACB + m\angle EAC = m\angle DEA,$$

which is a necessary and sufficient condition for XA to be tangent to ω'.

*This follows from the fact that T is the pole of the line BC with respect to ω. Thus, if M and M' are the two points of intersection of line TO with ω, and A' is the midpoint of BC; then $m\angle MAM' = 90$, and from the definition of pole and polar line, T, M, A' and M' are harmonic conjugates. Then it follows that AM and AM' are the internal and external bisectors of $\angle TAA'$, but AM is the angle bisector of $\angle BAC$, so AT is the reflection of AA' with respect to to the angle bisector AM.

O27. Let a, b, c be positive numbers such that $abc = 4$ and $a, b, c > 1$. Prove that

$$(a-1)(b-1)(c-1)(\frac{a+b+c}{3} - 1) \leq (\sqrt[3]{4} - 1)^4$$

Proposed by Marian Tetiva, Birlad, Romania

First solution by Aleksandar Ilic, Serbia.

Solution. Substitute $x = a-1$, $y = b-1$ and $z = c-1$. Now condition is that x, y, z are positive real numbers such that $(1+x)(1+y)(1+z) = 4$, and we have to prove inequality:

$$xyz \cdot \frac{x+y+z}{3} \leq (\sqrt[3]{4} - 1)^4.$$

From Newton's inequality we get

$$(xy + xz + yz)^2 \geq 3(xy \cdot xz + xy \cdot yz + xz \cdot yz) = 3xyz(x + y + z).$$

We will prove that $xy+xz+yz \leq 9(\sqrt[3]{4}-1)^2$ with equivalent condition $(x + y + z) + (xy + xz + yz) + xyz = 3$ using Lagrange multipliers. So, we examine symmetrical function $\Phi(x, y, z) = xy + xz + yz + \lambda(x + y + z + xy + xz + yz + xyz)$ by finding partial derivatives.

$$\Phi'_x(x, y, z) = y + z + \lambda(1 + y + z + yz) = 0 \quad \Rightarrow \quad (1 + x)(y + z) = -4\lambda$$

$$\Phi'_y(x, y, z) = x + z + \lambda(1 + x + z + xz) = 0 \quad \Rightarrow \quad (1 + y)(x + z) = -4\lambda$$

$$\Phi'_z(x, y, z) = x + y + \lambda(1 + x + y + xy) = 0 \quad \Rightarrow \quad (1 + z)(x + y) = -4\lambda$$

With some manipulations we get system:

$$(x - y)(z - 1) = 0, \quad (y - z)(x - 1) = 0, \quad (z - x)(y - 1) = 0.$$

So, we have either $x = y = z$ or say $x = y = 1$. These are only possible points for extreme values. In first case we have $x = y = z = \sqrt[3]{4} - 1$ and $xy + xz + yz = 9(\sqrt[3]{4} - 1)^2$. In case $x = y = 1$ we get $z = 0$ and $xy + xz + yz = 1 < 9(\sqrt[3]{4} - 1)^2$. Points on border are only with $x = 0$ or $x = 3$, and these are trivial for consideration.

Second solution by Zhao Bin, HUST, China.

Solution. Let $x = a - 1, y = b - 1, z = c - 1$, then we have $x, y, z > 0$ and

$$xyz + xy + yz + zx + x + y + z = 3. \qquad (2)$$

The inequality is equivalent to:

$$xyz(x + y + z) \leq 3\left(\sqrt[3]{4} - 1\right)^4.$$

Denote $S = xyz(x + y + z)$, by

$$(x + y + z)^4 \geq 27xyz(x + y + z).$$

We have

$$x + y + z \geq \sqrt[4]{27S},$$

also

$$xyz + \frac{\left(\sqrt[3]{4} - 1\right)^2}{3}(x + y + z) \geq 2\sqrt{\frac{\left(\sqrt[3]{4} - 1\right)^2}{3}S},$$

and

$$xy + yz + zx \geq \sqrt{3xyz(x + y + z)} = \sqrt{3S}.$$

Combining the above three inequalities with equation (1), we get

$$\left(1 - \frac{\left(\sqrt[3]{4} - 1\right)^2}{3}\right)\sqrt[4]{27S} + 2\sqrt{\frac{\left(\sqrt[3]{4} - 1\right)^2}{3}S} + \sqrt{3S} \leq 3.$$

Thus it is easy to get $S \leq \left(\sqrt[3]{4} - 1\right)^4$, and the problem is solved.

O28. Let ϕ be Euler's totient function. Find all natural numbers n such that the equation $\phi(\ldots(\phi(x))) = n$ (ϕ iterated k times) has solutions for any natural k.

Proposed by Iurie Boreico, Moldova

Solution by Ashay Burungale, India.

Solution. Restate the problem as: find all infinite sequences of positive integers $a_n, n \geq 0$ satisfying $\phi(a_n) = a_{n-1}$. If x is not a power of 2, $\phi(x)$ is divisible by at least as high a power of two as x. Unless x is of the form $2^a * p^b$ with $p = 3(\text{mod } 4)$ the power is strictly greater. Unless $p = 3$ or $b = 1$, $\phi(\phi(x))$ is divisible by a strictly larger power of 2 than x. If $\phi(x)$ is divisible by an odd prime, x is also divisible by a (possibly different) odd prime. Hence, if any a_n is not a power of 2, all subsequent terms are, and the power of 2 dividing a_i is non-increasing for $i \geq n$, hence is ultimately constant. Hence terms are ultimately of the form $2^a \cdot 3^b$ or $2^a \cdot p$ with $p > 3$ and $p = 3(\text{mod } 4)$. In the second case, the sequence must be

$$2^a \cdot p, 2^a \cdot (2p+1), 2^a \cdot (4p+3), 2^a \cdot (8p+7), \ldots$$

where $p, 2p+1, 4p+3, 8p+7\ldots$ are all prime. The p^{th} term will be $2^{p-1}(p+1) - 1 \equiv p + 1 - 1 = 0(\text{mod } p)$, thus not prime. Hence this case cannot arise. So the possible sequences are

i) $a_n = 2^n$.

ii) for each k, $a_n = 2^n$ if $n < k$, $a_n = 2^k \cdot 3^{n-k}$ if $n \geq k$.

In particular, the answer to the original form of the question is all numbers of the form $2^a \cdot 3^b$ except 3.

O29. Let $P(x)$ be a polynomial with real coefficients of degree n with n distinct real zeros $x_1 < x_2 < ... < x_n$. Suppose $Q(x)$ is a polynomial with real coefficients of degree $n-1$ such that it has only one zero on each interval (x_i, x_{i+1}) for $i = 1, 2, ..., n-1$. Prove that the polynomial $Q(x)P'(x) - Q'(x)P(x)$ has no real zero.

Proposed by Khoa Lu Nguyen, Massachusetts Institute of Technology

Solution by Aleksandar Ilic, Serbia.

Solution. For polynomials $P(x) = a(x-x_1)(x-x_2)\dots(x-x_n)$ and $Q(x) = b(x-y_1)(x-y_2)\dots(x-y_{n-1})$ we have interlacing zeros

$$x_1 < y_1 < x_2 < y_2 < x_3 < \cdots < y_{n-1} < x_n.$$

Consider rational function, which is defined on R except for the points $x_1, x_2, \dots, x_n$

$$f(x) = \frac{Q(x)}{P(x)} = \frac{b}{a} \cdot \frac{(x-y_1)(x-y_2)\dots(x-y_{n-1})}{(x-x_1)(x-x_2)\dots(x-x_n)}.$$

Let $R(x) = P'(x)Q(x) - P(x)Q'(x)$. In points $x = x_i$, we have $R(x) = P'(x_i)Q(x_i) \neq 0$, because x_i isn't root of polynomial $Q(x)$ and $P'(x)$ has only roots with multiplicity one.

Lema: If $f(x) = a(x-x_1)(x-x_2)\dots(x-x_n)$ is polynomial with degree n and distinct real zeros $x_1 < x_2 < \cdots < x_n$, then

$$f_1(x) = \frac{f(x)}{x-x_1}, f_2(x) = \frac{f(x)}{x-x_2}, \dots, f_n(x) = \frac{f(x)}{x-x_n}.$$

form a basis for the polynomials of degree $n-1$.

Proof: We have n polynomials, and it is enough to prove that they are linearly independent. Assume that for some real $\alpha_1, \alpha_2, \dots, \alpha_n$ we have

$$g(x) = \sum_{i=1}^{n} \alpha_i \cdot f_i(x) = 0.$$

For $x = x_k$ we get $g(x_k) = \alpha_k f_k(x_k) = 0$ and thus $\alpha_k = 0$ for every $k = \overline{1,n}$.

According to lema above if we write $P_k(x) = \frac{P(x)}{x-x_k}$ then

$$Q(x) = c_1P_1(x) + c_2P_2(x) + \cdots + c_nP_n(x).$$

Evaluate $Q(x)$ at roots of polynomial $P(x)$.

$$Q(x_k) = c_k P_k(x_k) = c_k(x_k - x_1)(x_k - x_2)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n).$$

So, sign of $Q(x_k)$ is $\operatorname{sgn}(c_k)(-1)^{n-k}$. Because of interlacing property of zeros, we have that $Q(x_k)$ alternate in sign or equivalently that c_k have the same sign.

Let's calculate first derivative of $f(x)$.

$$f'(x) = \left(\frac{Q(x)}{P(x)}\right)' = \left(\sum_{i=1}^{n} \frac{c_i}{x - x_i}\right)' = -\sum_{i=1}^{n} \frac{c_i}{(x - x_i)^2} \neq 0.$$

Thus the problem is solved.

O30. Prove that equation

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+...+\frac{1}{x_n^2}=\frac{n+1}{x_{n+1}^2}$$

has a solution in positive integers if and only of $n \geq 3$.

Proposed by Oleg Mushkarov, Bulgarian Academy of Sciences, Sofia

First solution by Li Zhou, Polk Community College.

Solution. If $n=1$, then the equation becomes $\frac{1}{x_1^2}=\frac{2}{x_2^2}$, which has no solution since $\sqrt{2}$ is irrational.

Consider next that $n=2$. then the equation becomes $(x_2x_3)^2+(x_1x_3)^2=3(x_1x_2)^2$. For $1 \leq i \leq 3$, write $x_i=3^{n_i}y_i$, where y_i is not divisible by 3. Wlog, assume that $n_1 \geq n_2$. Then

$$3^{2(n_2+n_3)}((y_2y_3)^2+3^{2(n_1-n_2)}(y_1y_3)^2)=3^{2(n_1+n_2)+1}(y_1y_2)^2. \tag{3}$$

Since 1 is the quadratic residue modulo 3, $(y_2y_3)^2+3^{2(n_1-n_2)}(y_1y_3)^2 \equiv 1,2 \pmod 3$. Hence the exponents of 3 in the two sides of (3) cannot equal.

Finally, consider $n \geq 3$. Starting from $5^2=4^2+3^2$, we get $\frac{1}{12^2}=\frac{1}{15^2}+\frac{1}{20^2}$ by dividing by $3^2 4^2 5^2$. Multiplying by $\frac{1}{12^2}$, we get

$$\begin{aligned}\frac{1}{12^4}=\frac{1}{12^2 15^2}+\frac{1}{12^2 20^2} &= \frac{1}{12^2 15^2}+(\frac{1}{15^2}+\frac{1}{20^2})\frac{1}{20^2}\\ &= \frac{1}{(12\cdot 15)^2}+\frac{1}{(15\cdot 20)^2}+\frac{1}{(20\cdot 20)^2}.\end{aligned}$$

Hence, $(x_1,x_2,x_3,x_4)=(12\cdot 15, 15\cdot 20, 20^2, 2\cdot 12^2)$ is a solution for $n=3$. Inductively, assume that $x_1,\dots,x_{n+1}$ are solutions to

$$\frac{1}{x_1^2}+\cdots+\frac{1}{x_n^2}=\frac{n+1}{x_{n+1}^2}$$

for some $n \geq 3$. Then

$$\frac{1}{x_1^2}+\cdots+\frac{1}{x_n^2}+\frac{1}{x_{n+1}^2}=\frac{n+2}{x_{n+1}^2},$$

completing the proof.

Second solution by Aleksandar Ilic, Serbia.

Solution. For $n = 1$, we get equation $\sqrt{2}x_1 = x_2$, and since $\sqrt{2}$ is irrational number - there are no solution in this case. For $n = 2$, we have equation $x_2^2x_3^2 + x_1^2x_3^2 = 3x_1^2x_2^2$ or equivalently $a^2 + b^2 = 3c^2$ with obvious substitution. We can assume that numbers a, b and c are all different from zero and that they are relatively prime, meaning $\gcd(a, b, c) = 1$. Square of an integer is congruent to 0 or 1 modulo 3, and hence both a and b are divisible by 3. Now, number c is also divisible by 3 - and we get contradiction.

For $n = 3$, we have at least one solution $(x_1, x_2, x_3, x_4) = (3, 3, 6, 4)$ or

$$\frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{6^2} = \frac{4}{4^2}.$$

For every integer $n > 3$, we can use solution for $n = 3$, and get:

$$\frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{6^2} + \underbrace{\frac{1}{4^2} + \cdots + \frac{1}{4^2}}_{n-3} = \frac{4}{4^2} + \frac{n-3}{4^2} = \frac{n+1}{4^2}.$$

Also solved by Ashay Burungale, India.